Laurent Series.

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Remark about two-sided series

Def $\sum_{n=-\infty}^{\infty} b_n$ converges if $\exists \lim_{N_1, N_2 \to \infty} \sum_{n=-N_1}^{N_2} b_n \iff$ both $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=1}^{\infty} b_{-n}$ converge.

If $\sum_{n=0}^{\infty} a_n w^n$ a power series with radius of convergence R,

then $\sum_{n=0}^{\infty} \frac{a_n}{(z-z_0)^n}$ converges locally uniformly when $\frac{1}{|z-z_0|} < R \iff |z-z_0| > \frac{1}{R}$.

$f(z) := \sum_{n=0}^{\infty} \frac{a_n}{(z-z_0)^n} \in \mathcal{A}\left(\{|z-z_0| > \frac{1}{R}\}\right)$ including ∞: $f(\infty) = a_0$.
removable singularity.

Let now $(a_n)_{n=-\infty}^{\infty}$, then

$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ converges in some annulus $\{R_1 < |z-z_0| < R_2\}$.
where R_2 - radius of convergence of $\sum_{n=0}^{\infty} a_n w^n$
$1/R_1$ - radius of convergence of $\sum_{n=1}^{\infty} a_{-n} w^n$.

$f \in \mathcal{A}(\{R_1 < |z-z_0| < R_2\})$ - locally uniform sum.

Note. Possible that $R_1 = 0$, $R_2 = \infty$.

Pierre Alphonse Laurent

Theorem (Laurent) Let $f \in \mathcal{A}(\underbrace{\{R_1 < |z-z_0| < R_2\}}_{=:\Omega})$

Then $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$.

The series converges locally uniformly in Ω.

$a_n = \frac{1}{2\pi i} \oint_{C_r} f(\zeta)(\zeta - z_0)^{-(n+1)} d\zeta$ for any $R_1 < r < R_2$,
$C_r := \{z + re^{it}\}$,

Proof

Observe: $\forall\ R_1 < r_1 < r_2 < R_2$, $C_{r_1} - C_{r_2} \sim 0$ in Ω.

Indeed, $z \notin \Omega \Rightarrow$ $|z| \geq R_2$: $n(C_{r_1}, z) = n(C_{r_2}, z) = 0$
$|z| \leq R_1$: $n(C_{r_1}, z) = n(C_{r_2}, z) = 1$.

So $\oint_{C_{r_1}} f(\zeta)(\zeta - z_0)^n d\zeta = \oint_{C_{r_2}} f(\zeta)(\zeta - z_0)^n d\zeta$ $\forall n \in \mathbb{Z}$ (can be negative).

since $f(\zeta)(\zeta - z_0)^n \in \mathcal{A}(\Omega)$.

Let us fix $z \in \Omega$ and r_1, r_2: $R_1 < r_1 < |z - z_0| < r_2 < R_2$

Notice $n(C_{r_2} - C_{r_1}, z) = 1$ $(n(C_{r_1}, z) = 0,\ n(C_{r_2}, z) = 1)$.

By Cauchy integral formula:

$$f(z) = \frac{1}{2\pi i}\oint_{C_{r_2}} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\oint_{C_{r_1}} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

Use Cauchy trick twice:

On $|\zeta - z_0| = r_2 > |z - z_0|$:

$$\frac{1}{\zeta - z} = \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{(\zeta - z_0)^{k+1}} \quad \left(\frac{|z-z_0|}{\underbrace{|\zeta - z_0|}_{\frac{|z-z_0|}{r_2}}} < 1 \right) \text{ – converges uniformly on } C_{r_2}$$

On $|\zeta - z_0| = r_1 < |z - z_0|$

$$-\frac{1}{\zeta - z} = \frac{1}{(z-z_0)\left(1 - \frac{\zeta - z_0}{z - z_0}\right)} = \sum_{k=1}^{\infty} \frac{(\zeta - z_0)^{k-1}}{(z - z_0)^k} \text{ – converges uniformly on } C_{r_1}.$$

$q = \frac{\zeta - z_0}{z - z_0}$, $|q| = \frac{r_1}{|z-z_0|} < 1$

Since multiplication by bounded $f(z)$ does not change uniform convergence, we get

$$f(z) = \sum_{k=0}^{\infty} \underbrace{\frac{1}{2\pi i}\oint_{C_{r_2}} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\,d\zeta}_{a_n} (z-z_0)^k + \sum_{k=1}^{\infty} \underbrace{\frac{1}{2\pi i}\oint_{C_{r_1}} f(\zeta)(\zeta - z_0)^{k-1}\,d\zeta}_{a_{-n}} (z-z_0)^{-k}$$

The series converges for every $z \in \Omega$, so locally uniformly ∎

Special case: $R_1 = 0$ – isolated singularity.

If $f \in \mathcal{A}(\Omega \setminus \{z_0\})$, $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ for $0 < |z - z_0| < \operatorname{dist}(z_0, \partial\Omega)$.

$(f \in \mathcal{A}(B(z_0, \operatorname{dist}(z_0, \partial\Omega)) \setminus \{z_0\}))$

Theorem.

1) z_0 is removable $\iff$ $\forall n < 0$, $a_n = 0$

2) z_0 is pole $\iff$ $\exists N \in \mathbb{N}$: $a_n = 0$ $\forall n < -N$.

3) z_0 is essential $\iff$ $\{n < 0 : a_n \neq 0\}$ is infinite.

Proof. 1) z_0 is removable $\iff$ $f \in \mathcal{A}(\Omega)$

So $a_{-n} = \frac{1}{2\pi i}\oint_{C_r} f(\zeta)(\zeta - z_0)^{n-1} d\zeta = 0$, by Cauchy $(n > 0)$.

Other direction: $a_n = 0$ $\forall n < 0$: $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ – analytic at z_0 ∎

2) already done: Theorem on characterization of poles.

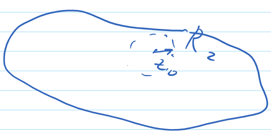
3) Essential ≠ pole or removable

So $\{n < 0 : a_n \neq 0\}$ is infinite $\iff$ essential ∎

Def. If z_0 is isolated singularity, $\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}$ is called singular part of Laurent decomposition.

$n = -1$: $a_n = \frac{1}{2\pi i}\oint_{C_r} f(\zeta)\,d\zeta = \operatorname{Res}_{z=z_0} f(z)$.

$r < \operatorname{dist}(z_0, \partial\Omega)$



<u>Def.</u> The <u>residue</u> of f at z_0 is defined as

$$R := \operatorname{Res}_{z=z_0} f(z) := \frac{1}{2\pi i} \oint_{C_r} f(z)\,dz,$$ where C_r is $\{z_0 + re^{it}\}$, the circle of radius r, centered at z_0 oriented counterclockwise,

$r < \operatorname{dist}(z_0, \partial\Omega)$.